On generalized universal irrational rotation algebras

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- Examples of strongly irreducible operators on infinite dimensional Hilbert spaces are from analytic Toeplitz operators, Cowen-Douglas operators, shift operators and so on.
- The main purpose of studying strongly irreducible operators is to extend the Jordan Canonical Form Theorem to infinite dimensional spaces.

Strongly irreducible operators relative to type II_1 factors

Relative strongly irreducible operator

Let M be a type II_1 factor, an operator $T \in M$ is said to be a strongly irreducible operator relative to M, if there exists no non-trivial idempotents in $\{T\}' \cap M$.

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Question

Are there any relative strongly irreducible operators in type II_1 factors?

$u + v$ in the hyperfinite type II_1 factor

Let R be the hyperfinite type II_1 factor. For an irrational number $\theta \in (0, 1)$, there are two unitary operators $u, v \in R$ satisfying the following properties:

- $R = \{u, v\}''$;

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$u + v$ is a relative strongly irreducible operator in R, i.e., there exists no nontrivial idempotents in $\{u + v\}' \cap R$.

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COROLLARY

The spectrum $\sigma(u + v)$ is connected.

The almost Mathieu operator

In mathematical physics, the almost Mathieu operator is given by

$$(H_{\lambda,\theta,\beta}u)(n) = u(n+1) + u(n-1) + 2\lambda\cos(2\pi(n\theta+\beta))u(n)$$

acting as a self-adjoint operator on the Hilbert space $l^2(\mathbb{Z})$. Here $\theta, \beta, \lambda \in \mathbb{R}$ are parameters.

Barry Simon raised several problems on the almost Mathieu operator. One problem (known as the "Ten martini problem" after Kac and Simon) conjectures that the spectrum of the almost Mathieu operator is a cantor set for all $\lambda \neq 0$ and irrational number θ.

The almost Mathiew operator $H_{\lambda,\theta,\beta}$ can be viewed as the operator

$$(u + \lambda e^{2\pi i\beta}v) + (u + \lambda e^{2\pi i\beta}v)^*$$

in the hyperfinite type II_1 factor R.

Spectrum of the almost Mathieu operator

Conjecture

If $a = (u + \lambda e^{2\pi i \beta} v) + (u + \lambda e^{2\pi i \beta} v)^*$, then $\sigma(a) =$ Cantor set,

where $\lambda \neq 0$, $vu = e^{2\pi i \theta} uv$, θ-irrational number.

Recently, Avila and Jitomirskaya affirmatively answered this conjecture (2009, Annals of Mathematics).

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Question

A natural question: What is the spectrum of $u + \lambda e^{2\pi i\beta} v$?

Spectrum of $u + \lambda v$

Theorem

The spectrum of $u + \lambda v$ is given by:

(1) $\sigma(u + v) = \overline{B(0,1)}$;

(2) $\sigma(u + \lambda v) = S^1, \lambda \in (0,1)$;

(3) $\sigma(u + \lambda v) = \lambda S^1, \lambda > 1$.

Spectral radius of $u + \lambda v$

Let $r(u+\lambda v)$ be the spectral radius of $u+\lambda v$. Then

$$r(u+\lambda v) = \lim_{n\to+\infty} ||(u+\lambda v)^n||^{\frac{1}{n}}$$

$$= \lim_{n\to+\infty} ||u^n(1+\lambda\omega)(1+\alpha\lambda\omega)\cdots(1+\alpha^{(n-1)}\lambda\omega)||^{\frac{1}{n}},$$

where $\omega = u^* v$ is a Haar unitary operator and $\alpha = e^{2\pi i\theta}$. Hence,

$$\begin{aligned} ||(u+\lambda v)^n||^{\frac{1}{n}} &= ||(1+\lambda\omega)(1+\alpha\lambda\omega)\cdots(1+\alpha^{(n-1)}\lambda\omega)||^{\frac{1}{n}} \\ &= ||(1+\lambda M_z)(1+\alpha\lambda M_z)\cdots(1+\alpha^{(n-1)}\lambda M_z)||^{\frac{1}{n}} \\ &= \left(\max_{z\in S^1} |(1+\lambda z)(1+\alpha\lambda z)\cdots(1+\alpha^{(n-1)}\lambda z)|\right)^{\frac{1}{n}} \\ &= \left(\max_{x\in[0,1]} \prod_{k=0}^{n-1}(1+\lambda^2+2\lambda\cos(2\pi(x+k\theta)))\right)^{\frac{1}{2n}} \end{aligned}$$

Spectral radius of $u + \lambda v$

Lemma

Suppose $0 < \lambda \leq 1$. For any $\varepsilon > 0$, there exists $x \in [0,1]$ and $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\left(\prod_{k=0}^{n-1}(1+\lambda^2+2\lambda\cos(2\pi(x+k\theta)))\right)^{\frac{1}{2n}} \geqslant 1-\varepsilon.$$

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Proof

Key point: Let $T : x \to x + \theta (\text{mod} 1)$. Then T is a measure preserving ergodic transformation of $[0,1]$. By Birkhoff's Ergodic theorem, for almost all $x \in [0,1]$, we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\ln(1+\lambda^2+2\lambda\cos(2\pi(x+k\theta)) = \int_0^1 \ln(1+\lambda^2+2\lambda\cos 2\pi x)dx = 0.$$

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- For all $x \in [0, 1]$, the points $\{e^{2\pi i(x+k\theta)} : 0 \leq k \leq n-1\}$ is almost uniformly distributed on $\mathbb{T}$.

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Corollary

$r(u + \lambda v) = 1, \forall 0 < \lambda \leq 1.$

Spectrum of $u+\lambda v$

Notice that, $u+v=u(1+u^*v)$. Since u^*v is a Haar unitary operator, $-1\in\sigma(u^*v)$. This implies that $u+v$ is not invertible and therefore $0\in\sigma(u+v)$. Observe that $\sigma(u+v)$ is rotation symmetric. Since $\sigma(u+v)$ is connected and $r(u+v)=1$, $\sigma(u+v)=\overline{B(0,1)}$.

For $0<\lambda<1$, $u+\lambda v=u(1+\lambda u^*v)$, so $0\notin\sigma(u+\lambda v)$. We can also show that $r((u+\lambda v)^{-1})=1$. Then

$$\sigma(u+\lambda v)=S^1.$$

For $\lambda>1$, consider $\lambda(\frac{1}{\lambda}u+v)$, we have $\sigma(u+\lambda v)=\lambda S^1$.

The von Neumann algebra and C^*-algebra generated by $u + v$

Theorem

For $\lambda > 0$, the von Neumann algebra generated by $u + \lambda v$ is R.

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For $\lambda > 0$ and $\lambda \neq 1$, $C^*(u + \lambda v) = C^*(u, v)$. However, $C^*(u + v)$ is a proper C^*-subalgebra of $C^*(u, v)$. Indeed, we have $\mathrm{dist}(u, C^*(u + v)) = 1$.

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Question

Is $C^*(u + v)$ $*$-isomorphic to $C^*(u, v)$?

Universal irrational rotation C^*-algebras

It is well known that the universal irrational rotation C^*-algebra $A_\theta = C^*(u, v)$ has the following properties (Rieffel [1981], Pimsner-Voiculescu [1980], Elliott-Evans [1993])

(ȧ) A_θ is simple.

(ḃ) There is a unique trace on A_θ.

(ċ) For every α in $(\mathbb{Z} + \mathbb{Z}\theta) \cap [0, 1]$, there exists a projection p in A_θ such that $\tau(p) = \alpha$.

(ḋ) $K_0(A_\theta) \cong \mathbb{Z} + \mathbb{Z}\theta$ and $K_1(A_\theta) \cong \mathbb{Z}^2$.

(ḟ) $A_\theta \cong A_\eta \Leftrightarrow \theta = \pm\eta \pmod{\mathbb{Z}}$.

(ġ) A_θ is an $A\mathbb{T}$-algebra and hence with real rank 0 and stable rank 1.

Generalized universal irrational rotation C^*-algebras

A generalized universal irrational rotation algebra $A_{\theta,\gamma} = C^*(x, w)$ is the universal C^*-algebra satisfying the following properties:

$$w^*w = ww^* = 1, \tag{1}$$

$$x^*x = \gamma(w), \tag{2}$$

$$xx^* = \gamma(e^{-2\pi i\theta}w), \tag{3}$$

$$xw = e^{-2\pi i\theta}wx, \tag{4}$$

where $\gamma(z) \in C(S^1)$ is a positive function.

- If $\gamma(z) \equiv 1$, then $A_{\theta,\gamma}$ is the universal irrational C^*-algebra,

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where $\gamma(z) \in C(S^1)$ is a positive function.

- If $\gamma(z) \equiv 1$, then $A_{\theta,\gamma}$ is the universal irrational C^*-algebra,
- If $\gamma(z) = |1+z|^2$, then $A_{\theta,\gamma}$ is $*$-isomorphic to $C^*(u+v)$.

A classical trace on generalized universal irrational rotation C^*-algebras

For $t \in [0,1]$, the pair $(e^{2\pi it}x, w)$ also satisfy (1)-(4). Thus there is an automorphism ρ_t of $A_{\theta,\gamma} = C^*(x,w)$ such that $\rho_t(x) = e^{2\pi it}x$ and $\rho_t(w) = w$. Define a map of $A_{\theta,\gamma} = C^*(x,w)$ into itself by

$$\Phi(a) = \int_0^1 \rho_t(a)dt.$$

Proposition

The map Φ is a faithful conditional expectation of $A_{\theta,\gamma} = C^*(x,w)$ onto $C^*(w)$. Furthermore, if ρ is the state on $C^*(w)$ induced by the Haar measure on $C(\mathbb{T})$, then $\tau = \rho \cdot \Phi$ is a classical faithful trace on $A_{\theta,\gamma} = C^*(x,w)$

Applying the GNS-construction to τ, $A_{\theta,\gamma} = C^*(x, w)$ is isomorphic to $C^*(u\gamma(v)^{1/2}, v) \subseteq A_\theta$ and the isomorphism takes x to $u\gamma(v)^{1/2}$ and w to v.

Tracial linear functionals on generalized universal irrational rotation C^*-algebras

Proposition

If μ is a complex regular Borel measure on $\mathbb{T}$ which satisfies that

$$\int_{\mathbb{T}} f(e^{-2\pi i\theta} z) d\mu(z) = \int_{\mathbb{T}} f(z) d\mu(z)$$

for all $f(z)$ in $\overline{\gamma(z)C(\mathbb{T})} \oplus \mathbb{C}1$ and let $\sigma(f) = \int_{\mathbb{T}} f(z) d\mu(z)$ for $f(z) \in C(\mathbb{T})$, then $\sigma \cdot \Phi$ is a bounded tracial linear functional on $A_{\theta,\gamma}$. Conversely, every bounded tracial linear functional on $A_{\theta,\gamma}$ is given in this way.

Tracial linear functionals on generalized universal irrational rotation C^*-algebras

Theorem

Suppose $\gamma(z)$ has finite zero points which can be divided into nonempty disjoint classes $A_1, \cdots, A_r$ in the following sense: z_1 and z_2 in A_j if and only if $z_2 = e^{2\pi i k\theta} z_1$ for some $k \in \mathbb{Z}$. Then the dimension of the space of tracial linear functionals on $A_{\theta,\gamma} = C^*(x, w)$ is $1 + \sum_{j=1}^{r}(|A_j| - 1)$, where $|A_j|$ is the number of elements in A_j.

Simple generalized universal irrational rotation C^*-algebras

Theorem

Let Y be the set of zero points of $\gamma(z)$ and let $\phi : \mathbb{T} \to \mathbb{T}$ be the rotation of the unit circle determined by θ, i.e., $\phi(z) = e^{2\pi i\theta} z$. Denote by $Orb(\xi) = \{\phi^n(\xi) : n \in \mathbb{Z}\}$ for $\xi \in \mathbb{T}$. Then the following properties are equivalent:

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1. $A_{\theta,\gamma}$ is simple;

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1. $A_{\theta,\gamma}$ is simple;
2. $A_{\theta,\gamma}$ has a unique tracial state;
3. $\phi^n(Y) \cap Y = \emptyset$ for all integer $n \neq 0$;
4. For each $\xi \in \mathbb{T}$, $Orb(\xi) \cap Y$ contains at most one point.

Riffel projections in generalized universal irrational rotation C^*-algebras

Theorem

If $m(\{z|\gamma(z)=0\})=0$, e.g, the zero points of $\gamma(z)$ is countable, then for every α in $(\mathbb{Z}+\mathbb{Z}\theta)\cap[0,1]$, there is a projection p in $A_{\theta,\gamma}=C^*(w,x)$ such that $\tau(p)=\alpha$.

K-groups of generalized universal irrational rotation C^*-algebras

Theorem

Let Y be the set of zero points of $\gamma(z)$. If Y is not empty, then $K_1(A_{\theta,\gamma}) \cong \mathbb{Z}$ and $K_0(A_{\theta,\gamma})$ is determined by the following splitting exact sequence

$$0 \to \mathbb{Z} \to K_0(A_{\theta,\gamma}) \to C(Y,\mathbb{Z}) \to 0.$$

We identify $A_{\theta,\gamma}$ with $C^*(u\gamma(v)^{1/2}, v) \subseteq C^*(u,v)$. Let $A = C^*(v)$ and

$$J_1 = \{f(v) : f \in C(\mathbb{T}) \text{ and } f(\lambda) = 0 \text{ for } \lambda \in Y\},$$

and let $J_2 = uJ_1u^*$. Then $A_{\theta,\gamma}$ is $*$-isomorphic to the covariance algebra $C^*(A,\Theta)$ for the partial automorphism $\Theta = (\mathrm{Ad}u, J_1, uJ_1u^*)$ of $C^*(v)$ in the sense of Ruy Exel.

K-groups of generalized universal irrational rotation C^*-algebras

- For a covariance algebra $C^*(A, \Theta)$ for the partial automorphism $\Theta = (\theta, J, \theta(J))$ of A, Ruy Exel proved the following generalized Pimsner-Voiculescu exact sequnece

$$\begin{array}{ccccc} K_0(J) & \xrightarrow{i_* - \theta_*^{-1}} & K_0(A) & \xrightarrow{i_*} & K_0(A_{\theta,\gamma}) \\ \uparrow & & & & \downarrow \\ K_1(A_{\theta,\gamma}) & \xleftarrow{i_*} & K_1(A) & \xleftarrow{i_* - \theta_*^{-1}} & K_1(J) \end{array}$$

Classification of simple generalized universal irrational rotation C^*-algebras

Theorem

Let $A_{\theta,\gamma}$ be a unital simple C^*-algebra. Then $A_{\theta,\gamma}$ is a unital simple $A\mathbb{T}$-algebra of real rank zero. In particular, $A_{\theta,\gamma}$ has tracial rank zero and stable rank one.

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- Decomposition rank and $\mathcal{Z}$-stability of Winter W.

Classification of simple generalized universal irrational rotation C^*-algebras

Theorem

Let θ_1 and θ_2 be two irrational numbers, γ_1 and $\gamma_2 \in C(\mathbb{T})$ be non-negative functions and let Y_i be the set of zeros of γ_i, $i = 1, 2$. Suppose that A_{θ_1,γ_1} and A_{θ_2,γ_2} are simple. Then $A_{\theta_1,\gamma_1} \cong A_{\theta_2,\gamma_2}$ if and only if the following hold:

$$\theta_1 = \pm\theta_2 mod(\mathbb{Z}) \quad C(Y_1, \mathbb{Z})/\mathbb{Z} \cong C(Y_2, \mathbb{Z})/\mathbb{Z}.$$

In particular, when γ_1 has only finitely many zeros, then $A_{\theta_1,\gamma_1} \cong A_{\theta_2,\gamma_2}$ if and only if $\theta_1 = \pm\theta_2 mod\mathbb{Z}$ and γ_2 has the same number of zeros.

$C^*(u+v)$

(a) $C^*(u+v)$ is a proper subalgebra of $\mathcal{A}_\theta = C^*(u,v)$. Indeed, $\inf\{||u-x|| : x \in C^*(u,v)\} = 1$.

(b) $C^*(u+v) \cong A_{\theta,\gamma}$ for $\gamma(z) = |1+z|^2$.

(c) $C^*(u+v)$ is a simple $A\mathbb{T}$-algebra of real rank zero and stable rank one.

(d) For every $\alpha \in (\mathbb{Z}+\mathbb{Z}\theta) \cap [0,1]$, there exists a projection p in $C^*(u+v)$ such that $\tau(p) = \alpha$.

(e) $K_0(C^*(u_\theta+v_\theta)) \cong \mathbb{Z}+\mathbb{Z}\theta$ and $K_1(C^*(u+v)) \cong \mathbb{Z}$. In particular, $C^*(u+v)$ is not $*$-isomorphic to $C^*(u,v)$.

(f) $C^*(u_\theta+v_\theta) \cong C^*(u_\eta+v_\eta) \Leftrightarrow \theta = \pm\eta(\mathrm{mod}\mathbb{Z})$.

Spectrum of $uf(v)$

Fang-Shi-Zhu, 2013

Let $f(z) \in L^\infty(S^1, m)$ and let $x = uf(v)$. Then the spectrum of x is given as follows:

1. If $f(v)$ is invertible, then $\sigma(uf(v)) = \Delta(f(v))S^1$.
2. If $f(v)$ is not invertible, then $\sigma(uf(v)) = \overline{\mathbb{B}(0, \Delta(f(v))}$.

Here $\Delta(f(v))$ is the Fuglede-Kadison determinant of $f(v)$.

Let M be a finite von-Neumann algebra with a faithful normal tracial state τ. The Fuglede-Kadison determinant $\triangle : M \to [0, \infty]$ is given by $\triangle(T) = \exp\{\tau(\ln|T|)\}, T \in M$, with $\exp\{-\infty\} := 0$.

Brown's Spectrum distribution

For an arbitrary element T in M, the function $\lambda \to \ln(\triangle(T-\lambda I))$ is subharmonic on $\mathbb{C}$, and its Laplacian:

$$d\mu_T(\lambda) = \frac{1}{2\pi}\nabla^2 \ln \triangle(T-\lambda I)$$

In the distribution sense, defines a probability measure μ_T on $\mathbb{C}$, called the Brown's spectral distribution or Brown's measure of T.

If T is normal, then μ_T is the trace τ composed with the spectral projection of T. If $M = M_n(\mathbb{C})$, then μ_T is the normalized counting measure $\frac{1}{n}(\delta_{\lambda_1} + \delta_{\lambda_2} + \ldots + \delta_{\lambda_n})$, where $\{\lambda_1, \ldots, \lambda_n\}$ are the eigenvalues of T repeated according to root multiplicity.

Brown's Spectrum distribution of $uf(v)$

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If $f(z) \in L^\infty(S^1, m)$, then the Brown measure of $uf(v)$ is the Haar measure on $\Delta(f(v))S^1$.

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Haagerup-Schultz, 2009

Let $T \in M$. If the support set of Brown's measure of T contains more than one point, then T has a nontrivial invariant subspace relative to M.